Bulk gravitational field and dark radiation on the brane in a dilatonic brane world

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We discuss the connection between dark radiation on the brane and the bulk gravitational field in a dilatonic brane world model proposed by Koyama and Takahashi where the exact solutions for the five-dimensional cosmological perturbations can be obtained analytically. It is shown that the dark radiation perturbation is related to the non-normalizable Kaluza-Klein (KK) mode of the bulk perturbations. For the de Sitter brane in the anti–de Sitter bulk, the squared mass of this KK mode is $2H^2$, where H is the Hubble parameter on the brane. This mode is shown to be connected to the excitation of a small black hole in the bulk in the long wavelength limit. The exact solution for anisotropic stress on the brane induced by this KK mode is found, which plays an important role in the calculation of cosmic microwave background radiation anisotropies in the brane world.

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I. INTRODUCTION

In the past few years, a lot of effort has been devoted to the investigation of the brane world scenario, where our Universe is a hypersurface, called a brane, embedded in a higher-dimensional bulk spacetime. Especially, models proposed by Randall and Sundrum have attracted much attention in the context of gravity and cosmology [1–3]. In their second model (RS model), a positive tension brane is embedded in five-dimensional anti–de Sitter (AdS) spacetime. The standard model particles are confined to the brane while the gravity can propagate in the bulk. An interesting feature of their model is that four-dimensional gravity can be recovered at low energy despite the infinite size of the extra dimension. It breaks the conventional idea that the extra dimension must be compact and small. The extension of the RS model to dilatonic brane worlds has been intensively investigated [4–15].

When we discuss the gravity on the brane, it is useful to derive the effective four-dimensional Einstein equation on the brane first developed by Shiromizu, Maeda, and Sasaki [16,5]. The effective four-dimensional Einstein equation includes the term $E_{\mu\nu}$, which is the electric part of five-dimensional Weyl tensor. This term is induced by the gravitational field in the bulk and carries the information in the bulk. In the RS model with AdS bulk spacetime, the $E_{\mu\nu}$ tensor can induce ''dark radiation'' on the brane in the homogeneous and isotropic background spacetime. It has been realized that the appearance of dark radiation on the brane is related to the existence of a black hole in the bulk.

Dark radiation provides interesting phenomena in the brane world cosmology. First, it modifies the expansion of the background Universe in the same way as done with usual radiation [17,18]. Second, it also gives important effects on cosmological perturbations on the brane. The cosmological perturbations in brane worlds have been actively discussed [19–26], and the possible impact on cosmic microwave background (CMB) anisotropies of dark radiation perturbation have been studied [27,28]. In this paper, we focus our attention on the dark radiation in cosmological perturbations.

The difficulty in the calculation of dark radiation perturbation is that it is no longer the radiation fluid once we consider the perturbation. This is because $E_{\mu\nu}$ could have a nontrivial component of an anisotropic stress. This renders distinguishable features for a dark radiation perturbation from an usual radiation fluid.

Because $E_{\mu\nu}$ is determined by the bulk gravitational field, it cannot be determined solely by the four-dimensional equations on the brane in general. Nevertheless, it is possible to know some features of this tensor by using constraint equations on the brane obtained by the four-dimensional Bianchi identity. In the background spacetime, the four-dimensional equations are sufficient to show that $E_{\mu\nu}$ induces the radiation fluid on the brane. In order to determine the amplitude of the energy density of dark radiation, the information in the bulk, that is, the mass of the black hole in the bulk, is needed. In the case of perturbations, it is impossible to determine the anisotropic component of $E_{\mu\nu}$ only from the four-dimensional equations. It is necessary to calculate the perturbations in the bulk.

The attempt to connect the dark radiation perturbation on the brane to bulk perturbations was made in Ref. [26]. However, in the RS model, it is impossible to find the analytic solutions for the bulk perturbations that properly satisfy the junction conditions at the brane. Thus it is difficult to analyze the precise relation between the dark radiation perturbation and bulk perturbations.

In this paper, we use a model given by Koyama and Takahashi [29,30]. This model is proposed in the context of an inflationary brane model induced by a bulk scalar field [31–36]. The great advantage of this model is that the five-dimensional cosmological perturbations can be solved analytically. Very recently, Kobayashi and Tanaka introduced a $(5+m)$-dimensional vacuum description of this model that makes the analysis of the cosmological perturbation simple [37]. They found complete sets of the solutions for bulk perturbations. The main purpose of this paper is to clarify the connection between the dark radiation perturbation and bulk perturbations in the bulk in this exactly solvable model.

The plan of this paper is as follows. In Sec. II, we briefly review the background spacetime. We then derive the four-dimensional effective Einstein equations and the equation of motion for the scalar field on the brane in Sec. III. In the dilatonic brane world, $E_{\mu\nu}$ contains the contribution from the bulk scalar field. In order to make it easy to compare our analysis with that in the RS model, we separate the contribution from the bulk scalar field in $E_{\mu\nu}$ and define a new tensor $F_{\mu\nu}$, which contains the information of the bulk gravitational fields. In Sec. IV, we discuss the "dark radiation" in cosmological perturbations on the brane. First, we find the dark-radiation-like solution for the constraint equations for $F_{\mu\nu}$ obtained by the four-dimensional Bianchi identity. We also calculate the exact solutions for $F_{\mu\nu}$ using the solutions of the bulk gravitational field obtained in Refs. [29,30,37]. Then, comparing these two results, it is possible to identify the bulk perturbation that induces dark-radiation-like contributions on the brane. In Sec. V, we discuss the connection between this bulk perturbation and the excitation of perturbatively small black holes in the bulk. In Sec. VI, we summarize the results and discuss the anisotropic stress induced by the dark radiation perturbation and its implication for CMB anisotropies.

II. BACKGROUND SPACETIME

We first review the background spacetime [29,30]. We start from the five-dimensional Einstein-Hilbert action with a bulk scalar field,

$$S=\int d^5x\sqrt{-g_5}\left(\frac{1}{2\kappa^2}R-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\Lambda(\varphi)\right)-\int d^4x\sqrt{-g_4}\lambda(\varphi), \tag{1}$$

where κ^2 is five-dimensional gravitational constant. The potential for the scalar field in the bulk and on the brane are taken to be exponential:

$$\kappa^2\Lambda(\varphi)=\left(\frac{\Delta}{8}+\delta\right)\lambda_0^2 e^{-2\sqrt{2}b\kappa\varphi}, \tag{2}$$

$$\kappa^2\lambda(\varphi)=\sqrt{2}\lambda_0 e^{-\sqrt{2}b\kappa\varphi}. \tag{3}$$

Here λ_0 is the energy scale of the potential, b is the dilaton coupling, and we defined

$$\Delta=4b^2-\frac{8}{3}. \tag{4}$$

We assume the Z_2 symmetry across the brane. This type of scalar field arises from a sphere reduction in M theory or string theory.

For $\delta=0$, the static brane solution was found [7]. The existence of the static brane requires tuning between the bulk potential and brane tension known as Randall-Sundrum tuning. It has been shown that for $\Delta\leqslant-2$, we can avoid the presence of the naked singularity in the bulk and also ensure the trapping of the gravity. The reality of the dilaton coupling requires $-8/3\leqslant\Delta$. Thus in the rest of the paper we shall assume $-8/3<\Delta<-2$. For $\Delta=8/3$, we recover the Randall-Sundrum model. The value of δ represents a deviation from the Randall-Sundrum tuning. This deviation drives an inflation on the brane.

The solution for background spacetime is found as

$$ds^2=e^{2W(y)}(-dt^2+e^{2\alpha(t)}\delta_{ij}dx^i dx^j+e^{2\sqrt{2}b\kappa\varphi(t)}dy^2),$$
$$\varphi(t,y)=\varphi(t)+\Xi(y). \tag{5}$$

The evolution equations for $\alpha(t)$ and $\varphi(t)$ are given by

$$\dot{\alpha}^2+\sqrt{2}b\kappa\dot{\varphi}\dot{\alpha}=\frac{1}{6}\kappa^2\dot{\varphi}^2-\frac{1}{3}\lambda_0^2\frac{\Delta+4}{\Delta}\delta e^{-2\sqrt{2}b\kappa\varphi}, \tag{6}$$

$$\ddot{\varphi}+(3\dot{\alpha}+\sqrt{2}b\kappa\dot{\varphi})\dot{\varphi}=-4\sqrt{2}b\kappa^{-1}\lambda_0^2\frac{\delta}{\Delta}e^{-2\sqrt{2}b\kappa\varphi}, \tag{7}$$

where the dot denotes the derivative with respect to t.

The solution for $\alpha(t)$ and $\varphi(t)$ can be easily found as

$$e^{\alpha(t)}=(H_0t)^{2/(3\Delta+8)}=(-H\eta)^{2/[3(\Delta+2)]}, \tag{8}$$

$$e^{\sqrt{2}b\kappa\varphi(t)}=H_0t=(-H\eta)^{(3\Delta+8)/[3(\Delta+2)]}, \tag{9}$$

where

$$H_0\equiv-\frac{3\Delta+8}{3(\Delta+2)}H,\quad H=-(\Delta+2)\sqrt{-\frac{\delta}{\Delta}}\lambda_0, \tag{10}$$

and a conformal time η is defined as

$$\eta=\int e^{-\alpha}dt=\frac{3\Delta+8}{3(\Delta+2)}H_0^{-2/(3\Delta+8)}t^{[3(\Delta+2)]/(3\Delta+8)}. \tag{11}$$

We should notice that power-law inflation occurs on the brane for $-8/3<\Delta<-2$.

The solutions for $W(y)$ and $\Xi(y)$ can be written as

$$e^{W(y)}=\mathcal{H}(y)^{2/[3(\Delta+2)]},\quad e^{\kappa\Xi(y)}=\mathcal{H}(y)^{2\sqrt{2}b/(\Delta+2)}, \tag{12}$$

where

$$\mathcal{H}(y)=\frac{\sinh Hy}{\sinh Hy_0},\quad \sinh Hy_0=\frac{1}{\sqrt{-1-\Delta/8\delta}}. \tag{13}$$

Here we assumed $\Delta/8+\delta<0$.

The above five-dimensional solution can be obtained by a coordinate transformation from the metric

$$ds^2=e^{2P(z)}(dz^2-d\tau^2+\delta_{ij}dx^i dx^j),\quad e^{\kappa\varphi(z)}=e^{3\sqrt{2}bP(z)}, \tag{14}$$

where

$$e^{P(z)}=(\sinh Hy_0)^{-2/3(\Delta+2)}(Hz)^{2/3(\Delta+2)}, \tag{15}$$

by

$$z=-\eta\sinh(Hy),$$

$$\tau = -\eta \cosh(Hy). \tag{16}$$

The metric (14) is often convenient because of its simplicity. [In Sec. V, we calculate the behavior of the five-dimensional Weyl tensor in the presence of a perturbatively small mass of black hole in the bulk not directly in Eq. (5) but in Eq. (14).]

The background equations on the brane, Eqs. (6) and (7), can be described by the four-dimensional Brans-Dicke (BD) theory with the action

$$S_{4,eff} = \frac{1}{2\kappa_4^2}\int d^4x \sqrt{-g_4}\left[\varphi_{BD}^{(4)}R - \frac{\omega_{BD}}{\varphi_{BD}}(\partial\varphi_{BD})^2\right] - \int d^4x \sqrt{-g_4} V_{eff}(\varphi_{BD}), \tag{17}$$

where

$$\varphi_{BD} = e^{\sqrt{2}b\kappa\varphi}, \quad \omega_{BD} = \frac{1}{2b^2},$$

$$\kappa_4^2 V_{eff}(\varphi_{BD}) = -\lambda_0^2 \delta \frac{\Delta+4}{\Delta}\frac{1}{\varphi_{BD}}. \tag{18}$$

III. EFFECTIVE EQUATIONS ON THE BRANE

In this section, we derive the effective gravitational equations on the brane using the covariant curvature formalism developed in Refs. [5,16]. Using the Gauss equation and Israel's junction condition, we obtain the induced four-dimensional Einstein equations on the brane as

$$^{(4)}G_{\mu\nu} = -{}^{(4)}\Lambda q_{\mu\nu} + \frac{2}{3}\kappa^2 T_{\mu\nu}^{(b)} - E_{\mu\nu}, \tag{19}$$

where

$$T_{\mu\nu}^{(b)} = D_\mu\varphi D_\nu\varphi - \frac{5}{8}q_{\mu\nu}(D\varphi)^2, \tag{20}$$

$$^{(4)}\Lambda = \frac{\kappa^2}{2}\left[\Lambda + \frac{\kappa^2}{6}\lambda^2 - \frac{1}{8}\left(\frac{d\lambda}{d\varphi}\right)^2\right] = \frac{\delta}{2}\lambda_0^2 e^{-2\sqrt{2}b\kappa\varphi}, \tag{21}$$

and

$$E_{\mu\nu} = {}^{(5)}C_{\mu\alpha\nu\beta}n^\alpha n^\beta. \tag{22}$$

We defined D_μ as the covariant derivative with respect to the induced metric on the brane. We note that four-dimensional cosmological constant $^{(4)}\Lambda$ is proportional to δ, which represents a deviation from the Randall-Sundrum tuning. The four-dimensional gravitational equation on the brane, Eq. (19), includes the projected Weyl tensor $E_{\mu\nu}$, which cannot be determined without solving the bulk dynamics in general. This term plays an essential role when we consider the cosmological perturbations in brane world models. Taking the divergence of the four-dimensional effective equations and using the four-dimensional Bianchi identity, we obtain the constraint equations for $E_{\mu\nu}$ as

$$D^\mu E_{\mu\nu} = \frac{2\kappa^2}{3}D^\mu T_{\mu\nu}^{(b)} - D^{\mu\,(4)}\Lambda q_{\mu\nu}. \tag{23}$$

Because $E_{\mu\nu}$ contains the contribution from the bulk scalar field, it is convenient to separate the contributions of the bulk scalar field from $E_{\mu\nu}$. We define

$$\begin{aligned}-E_{\mu\nu} = {} & \sqrt{2}b\kappa(D_\mu D_\nu\varphi - q_{\mu\nu}D^2\varphi) \\ & + 2b^2\kappa^2[D_\mu\varphi D_\nu\varphi - q_{\mu\nu}(D\varphi)^2] \\ & + \frac{\kappa^2}{3}\left(D_\mu\varphi D_\nu\varphi - \frac{1}{4}q_{\mu\nu}(D\varphi)^2\right) \\ & + \frac{6b^2}{\Delta}\lambda_0^2\delta e^{-2\sqrt{2}b\kappa\varphi}q_{\mu\nu} + F_{\mu\nu},\end{aligned} \tag{24}$$

We also rewrite the equation of motion for the scalar field on the brane as

$$D^2\varphi + \sqrt{2}b\kappa(D\varphi)^2 - 4\sqrt{2}\,\frac{b}{\kappa}\lambda_0^2\frac{\delta}{\Delta}e^{-2\sqrt{2}b\kappa\varphi} = F_\varphi. \tag{25}$$

From the traceless condition of $E_{\mu\nu}$, F_μ^μ are related to F_φ as

$$F_\mu^\mu = 3\sqrt{2}b\kappa F_\varphi. \tag{26}$$

The equations derived from the effective action (17) agree with Eqs. (24) and (25) with $F_{\mu\nu}=0$ and $F_\varphi=0$. Thus $F_{\mu\nu}$ and F_φ are expected to describe the contribution of KK modes. It should be noted that a similar decomposition of $E_{\mu\nu}$ was considered in Ref. [38].

Substituting the expression for $E_{\mu\nu}$, Eq. (24), and using the equation of motion for the scalar field, Eq. (25), we can rewrite the 4D Bianchi identity (23) as

$$D^\mu F_{\mu\nu} + \sqrt{2}b\kappa D^\mu\varphi F_{\mu\nu} = -\kappa^2 D_\nu\varphi F_\varphi. \tag{27}$$

In general, this constraint equation is not sufficient to completely determine the behavior of $F_{\mu\nu}$ and F_φ on the brane.

IV. DARK RADIATION IN COSMOLOGICAL PERTURBATIONS

In this section, we consider the dark radiation in cosmological perturbations on the brane. Since dark radiation corresponds to scalar-type perturbations, we restrict our attention to cosmological perturbations of this type throughout the paper. It is assumed that $F_{\mu\nu}=0$ and $F_\varphi=0$ for the background spacetime in Koyama-Takahashi model.

First, we show that dark radiation appears as a solution of the constraint equations for $\delta F_{\mu\nu}$ and δF_φ on the brane, Eq. (27), at large scales in Sec. IV A. $\delta F_{\mu\nu}$ and δF_φ have four independent variables for scalar perturbations. We also show that two of the four variables cannot be determined by their constraint equations (27).

Next, we calculate the exact solution of $\delta F_{\mu\nu}$ and δF_φ using the solutions for the five-dimensional perturbed Einstein equations obtained by Refs. [29,30,37] in Sec. IV B.

We then investigate the relation between dark radiation and bulk perturbations.

A. View from the brane

Here we consider dark radiation as a solution of the constraint equations for $\delta F_{\mu\nu}$ and δF_{φ} on the brane, Eq. (27). First, we expand $\delta F_{\mu\nu}$ in terms of the scalar harmonics as

$$\delta F_{tt}=\delta\rho_F Y,$$

$$\delta F_{ti}=e^{\alpha}\delta q_F Y_i,$$

$$\delta F_{ij}=e^{2\alpha}\left(\frac{1}{3}(\delta\rho_F+3\sqrt{2}b\kappa\delta F_{\varphi})Y\delta_{ij}+\delta\pi_F Y_{ij}\right), \tag{28}$$

where $Y(k,x)\propto e^{ikx}$ is the normalized scalar harmonics and the vector Y_i and traceless tensor Y_{ij} are constructed from Y as $Y_i=-k^{-1}Y_{,i}$, $Y_{ij}=k^{-2}Y_{,ij}+\delta_{ij}Y/3$.

The four-dimensional perturbed Einstein equations and the equation for the scalar field are not closed but include four variables $\delta\rho_F$, δq_F, $\delta\pi_F$, and δF_{φ}. The concrete forms of these equations are presented in Appendix A. On the other hand, there are constraint equations on $\delta F_{\mu\nu}$ and δF_{φ} [Eq. (27)]. For scalar-type perturbations, these become two equations as follows:

$$(\partial_t+4\dot{\alpha}+\sqrt{2}b\kappa\dot{\varphi})\delta\rho_F-ke^{-\alpha}\delta q_F=0, \tag{29}$$

$$(\partial_t+4\dot{\alpha}+\sqrt{2}b\kappa\dot{\varphi})\delta q_F +ke^{-\alpha}\left(\frac{2}{3}\delta\pi_F-\frac{1}{3}\delta\rho_F-\sqrt{2}b\kappa\delta F_{\varphi}\right)=0. \tag{30}$$

Here we used the condition, Eq. (26). At large scales $ke^{-\alpha}/H\to 0$, we can neglect the term proportional to δq_F in Eq. (29). The solution of $\delta\rho_F$ is given by $\delta\rho_F=\delta C e^{-4\alpha-\sqrt{2}b\kappa\varphi}$. This corresponds to dark radiation (hereafter we call this dark radiation although this does not behave as a radiation for $b\neq 0$). It can be checked that the integration constant δC is related to the perturbatively small black hole mass in the bulk (see Sec. V). On the other hand, we cannot determine $\delta\pi_F$ because it is dropped from Eq. (30) for $ke^{-\alpha}/H\to 0$. This uncertainty prevents us from predicting CMB anisotropies in brane world models [27,28]. This issue is discussed in the Sec. VI.

Since there are only two constraint equations, two of the four variables cannot be determined. On the other hand, there are two physical degrees of freedom in the bulk for scalar-type perturbations. One of them corresponds to the scalar field and the other to the graviscalar. To investigate the relation between dark radiation and bulk perturbations, we need to obtain the exact solutions for $\delta F_{\mu\nu}$ and δF_{φ}. This can be achieved only when we solve the bulk gravitational field and determine the behavior of the two unknown variables.

B. View from the bulk

In the preceding section, we showed that we must solve the bulk gravitational field in order to completely determine the contributions of $F_{\mu\nu}$ and F_{φ} to cosmological perturbations on the brane. Here, we first summarize the two independent solutions of five-dimensional Einstein equations for scalar perturbations obtained in Ref. [37] in Sec. IV B 1. Using these solutions, we then calculate and investigate the behavior of the contributions of $F_{\mu\nu}$ and F_{φ} to cosmological perturbations on the brane in Sec. IV B 2. Finally we discuss their relation to dark radiation in Sec. IV B 3.

1. Solutions of the bulk gravitational field

Here, we present the two independent solutions of five-dimensional Einstein equations for scalar perturbations obtained in Ref. [37]. The perturbed metric and scalar field are given by

$$\begin{aligned} ds^2=e^{2W(y)}\{&e^{2\sqrt{2}b\kappa\varphi(t)}(1+2NY)dy^2+2AYdtdy \\ &-(1+2\Phi Y)dt^2+e^{2\alpha(t)}[(1+2\Psi Y)\delta_{ij}dx^i dx^j \\ &+2EY_{ij}dx^i dx^j+2BY_i dx^i dt+2CY_i dx^i dy]\}, \\ \varphi=\varphi(t)&+\Xi(y)+\delta\varphi Y. \end{aligned} \tag{31}$$

We note that any gauge condition is not imposed here.

Under a scalar gauge transformation,

$$t\to\overline{t}=t+\xi^t Y,$$

$$y\to\overline{y}=y+\xi^y Y,$$

$$x^i\to\overline{x^i}=x^i+\xi^S Y^i, \tag{32}$$

the metric variables transform as

$$N\to\overline{N}=N-\xi^{y\,\prime}-W'\xi^y-\sqrt{2}b\kappa\dot{\varphi}\xi^t,$$

$$A\to\overline{A}=A+\xi^{t\,\prime}-e^{2\sqrt{2}b\kappa\varphi}\dot{\xi}^y,$$

$$C\to\overline{C}=C+e^{-2\alpha+2\sqrt{2}b\kappa\varphi}k\xi^y-\xi^{S\,\prime},$$

$$\Phi\to\overline{\Phi}=\Phi-W'\xi^y-\dot{\xi}^t,$$

$$B\to\overline{B}=B-e^{-2\alpha}k\xi^t-\dot{\xi}^S,$$

$$E\to\overline{E}=E+k\xi^S,$$

$$\Psi\to\overline{\Psi}=\Psi-\frac{1}{3}k\xi^S-W'\xi^y-\dot{\alpha}\xi^t,$$

$$\delta\varphi\to\overline{\delta\varphi}=\delta\varphi-\dot{\varphi}\xi^t-\Xi'\xi^y, \tag{33}$$

where the prime denotes the derivative with respect to y. In our background spacetime, the gauge fixing condition imposed in Ref. [37] corresponds to

$$N=\sqrt{2}b\kappa\delta\varphi,\quad A=C=0. \tag{34}$$

These conditions do not fix the gauge completely. We use this remaining degree of freedom to keep the brane location unperturbed at $y=y_0$. Then, the boundary conditions on the brane for all the remaining variables become Neumann boundary conditions:

$$\partial_y N|_{y_0}=\partial_y \Phi|_{y_0}=\partial_y \Psi|_{y_0}=\partial_y E|_{y_0}=\partial_y B|_{y_0}=0. \quad (35)$$

All variables can be expanded by the same mode functions in the y direction as

$$\Phi=\Phi_0(t)\psi_0(y)+\sum \Phi_m(t)\psi_m(y), \ldots, \quad (36)$$

where ψ_0 is constant, and

$$\psi_m(y)=c(\sinh Hy)^{1/2+\mu}B^{-1/2-\mu}_{-1/2+i\nu}(\cosh Hy),$$

$$B^{\beta}_{\gamma}(\cosh Hy)=Q^{1/2-\mu}_{-1/2+i\nu}(\cosh Hy_0)P^{\beta}_{\gamma}(\cosh Hy) - P^{1/2-\mu}_{-1/2+i\nu}(\cosh Hy_0)Q^{\beta}_{\gamma}(\cosh Hy), \quad (37)$$

$$\mu=-\frac{1}{(\Delta+2)}, \quad (38)$$

$$\nu(m)=\sqrt{\frac{m^2}{H^2}-\mu^2}, \quad (39)$$

where c is a normalization constant. The first and second terms in Eq. (36) represent the zero and KK modes, respectively. m^2 represents the squared KK mass for observers on the four-dimensional brane. There is a mass gap $\delta m=\mu H$ between the zero mode and the KK continuum. The modes with $0<m<\mu H$ are not normalizable. P^{α}_{β} and Q^{α}_{β} are associated Legendre functions.

We now turn to the mode functions in the t direction. For the zero mode, we have another gauge degree of freedom. As is evident from Eq. (33), gauge transformation satisfying $\xi^y=0$ and $\xi^{t\prime}=\xi^{S\prime}=0$ do not disturb the conditions (34). We can use this degree of freedom to set $B=E=0$, because the solutions do not depend on y for the zero mode. The solutions are given by

$$N_0=\sqrt{2}b\kappa\delta\varphi_0 =c_0\frac{1}{3}\frac{\Delta+2}{\Delta+3}\left(\rho_\mu-\frac{3\Delta+8}{\Delta+4}\rho_{\mu-2}\right),$$

$$\Psi_0=-c_0\frac{2}{3}\frac{\Delta+2}{\Delta+3}\left(\rho_\mu+\frac{1}{\Delta+4}\rho_{\mu-2}\right),$$

$$\Phi_0=-\Psi_0-N_0, \quad (40)$$

where c_0 is a constant and ρ_α is defined as

$$\rho_\alpha(\eta)=(-k\eta)^\mu H_\alpha(-k\eta), \quad (41)$$

where H_α is an arbitrary linear combination of Hankel functions $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$. We note that the number of physical degrees of freedom is one for the zero mode. This solution is already obtained by Koyama and Takahashi [29,30].

The solutions for the KK modes in the gauge condition (34) are obtained as

$$N_m=\sqrt{2}b\kappa\delta\varphi_m =-\frac{1}{3}\frac{3\Delta+8}{\Delta+4}\{c_1[(2\mu-1)k\eta\rho_{i\nu-1} +(i\nu+\mu)(i\nu+\mu-1)\rho_{i\nu}]+2c_2\rho_{i\nu}\}, \quad (42)$$

$$\Phi_m=c_1(-k\eta)^2\rho_{i\nu}+N_m, \quad (43)$$

$$\Psi_m=-\frac{1}{3}c_1(-k\eta)^2\rho_{i\nu}, \quad (44)$$

$$E_m=c_1(-k\eta)^2\rho_{i\nu}-\frac{2}{\Delta+4}c_1[(2\mu-1)k\eta\rho_{i\nu-1} +(i\nu+\mu)(i\nu+\mu-1)\rho_{i\nu}]+\frac{3\Delta+8}{\Delta+4}c_2\rho_{i\nu}, \quad (45)$$

$$B_m=2c_1e^{-\alpha}k\eta[(i\nu+\mu-1)\rho_{i\nu}+k\eta\rho_{i\nu-1}]. \quad (46)$$

We should note that the solution obtained in Ref. [30] is a particular solution where c_1 and c_2 are related (see Appendix B).

2. Solutions for $F_{\mu\nu}$ and F_φ

Next, we calculate the solutions for $\delta F_{\mu\nu}$ and δF_φ, using the solutions of the bulk gravitational field summarized above. The above two solutions are obtained in the Gaussian-normal gauge condition with respect to the brane. After a gauge transformation to the longitudinal gauge (see Appendix B), we substitute the solutions projected on the brane into the four-dimensional perturbed Einstein equations in Appendix A [Eqs. (A2), (A4), (A5), and (A7)]. Then we obtain $\delta F_{\mu\nu}$ and δF_φ as

$$\delta\rho_F=-\tilde{c}_1e^{-2\alpha}\rho_{i\nu}(\eta)\psi_m(y_0), \quad (47)$$

$$\delta q_F=\tilde{c}_1e^{-2\alpha}(k\eta)^{-1}[(i\nu+\mu-1)\rho_{i\nu}(\eta) +k\eta\rho_{i\nu-1}(\eta)]\psi_m(y_0), \quad (48)$$

$$\delta\pi_F=\tilde{c}_1(i\nu+\mu-1)e^{-2\alpha}\left[\frac{1}{i\nu-1}\left(\rho_{i\nu}(\eta) +\frac{\mu}{i\nu+\mu-1}\rho_{i\nu-2}(\eta)\right)-2\frac{i\nu+\mu}{\Delta+4}\frac{\rho_{i\nu}(\eta)}{k^2\eta^2}\right]\psi_m(y_0) +\tilde{c}_2\frac{3\Delta+8}{\Delta+4}e^{-2\alpha}(k\eta)^{-2}\rho_{i\nu}(\eta)\psi_m(y_0), \quad (49)$$

$$\delta F_\varphi=\tilde{c}_1\sqrt{2}b\kappa^{-1}(i\nu+\mu-1)e^{-2\alpha}\times\left[2\frac{i\nu+\mu}{\Delta+4}\frac{\rho_{i\nu}(\eta)}{k^2\eta^2}-\frac{\mu}{(i\nu+\mu-1)(i\nu-1)}\times[\rho_{i\nu}(\eta)+\rho_{i\nu-2}(\eta)]\right]\psi_m(y_0)+\tilde{c}_2\frac{4\sqrt{2}b\kappa^{-1}}{\Delta+4}e^{-2\alpha}(k\eta)^{-2}\rho_{i\nu}(\eta)\psi_m(y_0), \quad (50)$$

where

$$\tilde{c}_1=(i\nu+\mu)(i\nu-\mu)k^2c_1=-\frac{m^2}{H^2}k^2c_1, \quad (51)$$

$$\tilde{c}_2=(i\nu+\mu)(i\nu-\mu)k^2c_2=-\frac{m^2}{H^2}k^2c_2. \quad (52)$$

As expected, $\delta F_{\mu\nu}$ and δF_φ vanish for $m^2=0$.

3. Dark radiation and bulk perturbation

In Sec. IV B 2, we calculated the contributions of the bulk perturbation to $F_{\mu\nu}$ and F_φ on the brane. Here, we discuss their relation to dark radiation. As mentioned before, there are two physical degrees of freedom in the bulk for scalar perturbations. One of them corresponds to the scalar field and the other to the graviscalar. Since the c_2 component of $\delta\rho_F$ vanishes, it is expected that the solution of c_1 includes dark radiation at large scales and thus corresponds to the graviscalar. This can be explicitly shown if we take $i\nu+\mu-1=0$ and a linear combination of Hankel functions:

$$H_\alpha^{(1)}(-k\eta)+e^{-2i\alpha\pi}H_\alpha^{(2)}(-k\eta)=2e^{-i\alpha\pi}J_{-\alpha}, \quad (53)$$

such that $H_\alpha(-k\eta)\propto(-k\eta)^{-\alpha}$ for $-k\eta\to 0$. In this case, the above solutions for $\delta F_{\mu\nu}$ and δF_φ becomes

$$\delta\rho_F=-(1-2\mu)c_1e^{-2\alpha}\rho_{1-\mu}\psi_m(y_0)\propto-\tilde{c}_1e^{-4\alpha-\sqrt{2}b\kappa\varphi}, \quad (54)$$

$$\delta q_F\propto-\tilde{c}_1\frac{-k\eta}{2\mu}e^{-4\alpha-\sqrt{2}b\kappa\varphi}, \quad (55)$$

$$\delta\pi_F\propto-\tilde{c}_1\frac{(-k\eta)^2}{4\mu(\mu+1)}e^{-4\alpha-\sqrt{2}b\kappa\varphi}+\tilde{c}_2\frac{3\Delta+8}{\Delta+4}\eta^{-2}e^{-4\alpha-\sqrt{2}b\kappa\varphi}, \quad (56)$$

$$\delta F_\varphi\propto\tilde{c}_1\sqrt{2}b\kappa^{-1}e^{-4\alpha-\sqrt{2}b\kappa\varphi}+\tilde{c}_2\frac{4\sqrt{2}b\kappa^{-1}}{\Delta+4}\eta^{-2}e^{-4\alpha-\sqrt{2}b\kappa\varphi}, \quad (57)$$

for $-k\eta\to 0$. The time dependence of $\delta\rho_F$ coincides with that of the solution of the constraint equation (29) for large scales. It should be noted that the condition $i\nu+\mu-1=0$ can be written as

$$m^2=(2\mu-1)H^2(<\mu^2H^2). \quad (58)$$

This is quite an interesting result. The dark radiation corresponds to a non-normalizable KK mode. For the RS model $b=0$, the mass squared becomes $2H^2$.

It should be also emphasized that the behavior of $\delta\pi_F$, which corresponds to dark radiation, is obtained here [Eq. (56)]. This variable cannot be known by the constraint equation (30) because $\delta\pi_F$ is dropped for $k\to 0$. As mentioned above, this uncertainty prevents us from predicting CMB anisotropies in brane world models [27,28]. This issue is discussed later in the Sec. VI.

V. BLACK HOLE IN THE BULK AND A KK MODE

In the preceding section, we showed that dark radiation corresponds to a non-normalizable KK mode of cosmological perturbations. Here, we discuss the connection between this KK mode and the black hole in the bulk when the black hole mass is perturbatively small.

There is a black hole solution for a bulk scalar field with an exponential potential, which coincides with the background spacetime of the Koyama-Takahashi model when the black hole mass vanishes. We first review this black hole solution in Sec. V A. We also calculate the behavior of $E_{\mu\nu}$ in the case where the black hole mass is perturbatively small. We also calculate the perturbation of $E_{\mu\nu}$ using the solutions of the perturbed five-dimensional Einstein equations for $i\nu+\mu-1=0$. It is shown that the asymptotic behavior of E_{tt} in the bulk coincides with that which originates from the black hole in the bulk.

A. Black hole solution with a bulk scalar field

Here we review a black hole solution with a bulk scalar field that has the exponential potential in the bulk, Eq. (2). When the black hole mass vanishes, this solution coincides with the background spacetime of the Koyama-Takahashi model.

We can find a static solution for the bulk with vanishing cosmological constant as [10,13,39]

$$ds^2=-h(R)dT^2+\frac{R^{3\Delta+8}}{h(R)}dR^2+R^2\delta_{ij}dx^idx^j, \quad (59)$$

$$\varphi=3\sqrt{2}b\kappa^{-1}\ln(R) \quad (60)$$

where

$$h(R)=\tilde{\lambda}_0^2R^2-CR^{6b^2-2} \quad (61)$$

and C is an arbitrary constant that is related to black hole mass. Here we defined

$$\tilde{\lambda}_0^2 = \frac{\lambda_0^2}{18}\left(1+\delta\frac{8}{\Delta}\right). \tag{62}$$

For $b=0$, this solution becomes AdS-Schwartzshild. The Friedmann equation on the brane with the tension [Eq. (3)] is obtained as [13]

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{4}{9}\left(\frac{\delta}{-\Delta}\right)\lambda_0^2 R^{-(3\Delta+8)} + CR^{-(3\Delta+16)/2}, \tag{63}$$

where dot is the derivative with respect to cosmic time on the brane.

For $C=0$, the background spacetime (14) can be obtained from this metric by a coordinate transformation,

$$R=\left(-\frac{3}{2}\tilde{\lambda}_0(\Delta+2)z\right)^{2/3(\Delta+2)}, \tag{64}$$

$$T=\frac{\tau}{\tilde{\lambda}_0}. \tag{65}$$

When there is a perturbatively small mass of black hole in the bulk, the metric (14) is modified as

$$ds^2 = e^{2P(z)}\{[1+\delta C f(z)]dz^2 - [1-\delta C f(z)]d\tau^2 + \delta_{ij}dx^i dx^j\}, \tag{66}$$

where

$$f(z)=\frac{1}{\tilde{\lambda}_0^2}R(z)^{6b^2-4}, \tag{67}$$

and δC is perturbed black hole mass. It is noted that this modification cannot be regarded as a perturbation at $z\to\infty$ or $R\to 0$. We focus our attention to the region sufficiently far from the black hole so that the above modification can be treated as a perturbation. We can easily calculate the five-dimensional Weyl tensor in this coordinate system as

$$C_{\tau z\tau z} = -\frac{1}{4}\delta C R(z)^2 \partial_z^2 f(z). \tag{68}$$

By a coordinate transformation (16), this is related to the Weyl tensor in our background spacetime (5) as

$$C_{\eta y\eta y} = \eta^2 H^2 C_{\tau z\tau z}. \tag{69}$$

Finally, we obtain the behavior of E_{tt}:

$$\delta E_{tt} = \frac{9}{8}\Delta\,\delta C e^{-4\alpha-\sqrt{2}b\kappa\varphi}\left(\frac{\sinh Hy}{\sinh Hy_0}\right)^{2\mu-1}. \tag{70}$$

Other components can be obtained by the homogeneity and isotropy of three-dimensional spatial coordinates and the condition $E^{\mu}_{\mu}=0$.

B. Perturbations of $E_{\mu\nu}$ in the bulk

Here we calculate perturbations of $E_{\mu\nu}$ in the bulk, using the solutions of the five-dimensional perturbed Einstein equations for $i\nu+\mu-1=0$ at large scales. In Appendix C, we present the perturbation formula for the five-dimensional Weyl tensor in the Gaussian normal gauge. Substituting the solutions of the bulk gravitational field, we get

$$\delta E_{tt} = -\frac{\Delta}{4}c_1 k^2 e^{-2\alpha}\rho_{1-\mu}\left(\frac{1}{H^2}\partial_y^2\psi_m(y)-\psi_m(y)\right). \tag{71}$$

Here we took $i\nu+\mu-1=0$ and $-k\eta\to 0$. Using the solution for $\psi_m(y)$, the y dependence of E_{tt} can be evaluated as

$$\frac{1}{H^2}\partial_y^2\psi_m(y)-\psi_m(y) = -\frac{2\mu}{1-2\mu}(\sinh Hy)^{1/2+\mu}B^{3/2-\mu}_{1/2-\mu}(\cosh Hy). \tag{72}$$

On the brane, due to the junction condition, we can show that

$$B^{3/2-\mu}_{1/2-\mu}(\cosh Hy_0) = (1-2\mu)B^{-1/2-\mu}_{1/2-\mu}(\cosh Hy_0). \tag{73}$$

Then E_{tt} on the brane is given by

$$\delta E_{tt}(y_0) = \frac{\Delta}{2}c_1 k^2 e^{-2\alpha}\mu\rho_{1-\mu}\psi_m(y_0). \tag{74}$$

Comparing this solution with Eq. (70), it is possible to express c_1 by the black hole mass δC as

$$c_1 k^2 = \frac{9}{4}\frac{(-H\eta)^{2\mu-1}}{\mu\rho_{1-\mu}}\psi_m(y_0)^{-1}\delta C. \tag{75}$$

We note that the left-hand side does not depend on time for $-k\eta\to 0$. We can also rewrite the induced four-dimensional Einstein equation (19) in accordance with the Friedmann equation (63) as

$$^{(4)}G_{\mu\nu} = \frac{8}{3\Delta}{}^{(4)}\Lambda q_{\mu\nu} - \tilde{E}_{\mu\nu}, \tag{76}$$

where

$$-\tilde{E}_{\mu\nu} = -E_{\mu\nu} + \frac{2}{3}\kappa^2 T^{(b)}_{\mu\nu} - \frac{2b^2}{\Delta}\lambda_0^2\delta e^{-2\sqrt{2}b\kappa\varphi}q_{\mu\nu}. \tag{77}$$

If we substitute the solutions for $i\nu+\mu-1=0$ and then use the relation (75), $\tilde{E}_{tt}$ becomes

$$-\delta\tilde{E}_{tt} = 3\,\delta C e^{-4\alpha-\sqrt{2}b\kappa\varphi}. \tag{78}$$

Clearly, this corresponds to the dark radiation term in Eq. (63). Finally we investigate the y dependence of E_{tt}. For large Hy, the y dependence of E_{tt} behaves as

$$\frac{1}{H^2}\partial_y^2\psi_m(y)-\psi_m(y)\propto(\sinh Hy)^{2\mu-1}. \qquad (79)$$

This behavior is precisely the same as δE_{tt} derived from the black hole solution. Thus the correspondence is held also in the bulk.

In the perturbation solutions, there is also an anisotropic part of $E_{\mu\nu}$. The result for $i\nu+\mu-1=0$ is

$$\delta\pi_E=-\frac{8}{3\Delta}\frac{(-k\eta)^2}{4\mu(\mu+1)}\delta E_{tt}, \qquad (80)$$

where $\delta\pi_E$ is defined in the same way with $\delta\pi_F$.

VI. SUMMARY AND DISCUSSION

In this paper, we discussed the connection between dark radiation and bulk perturbations in a dilatonic brane world based on a model proposed by Koyama and Takahashi [29,30]. We first derived the four-dimensional effective Einstein equations on the brane developed in Refs. [5,16]. We separated the contributions of the bulk scalar field from $E_{\mu\nu}$. Then the four-dimensional effective theory becomes the BD theory with the corrections given by $F_{\mu\nu}$ and F_φ. We then considered dark radiation in cosmological perturbations on the brane. The perturbed Einstein equations include the four variables $\delta\rho_F$, δq_F, $\delta\pi_F$, and δF_φ, which carry the information in the bulk. There are two constraint equations obtained from the four-dimensional Bianchi identity. We showed that the dark radiation appears as a solution for the constraint equations at large scales.

We can derive a complete set of the solutions for $\delta\rho_F$, δq_F, $\delta\pi_F$, and δF_φ only when we solve the bulk gravitational fields, which have two physical degrees of freedom: the scalar field perturbation and the graviscalar. We calculated $\delta\rho_F$, δq_F, $\delta\pi_F$, and δF_φ on the brane using these two independent solutions of the bulk perturbations obtained in Refs. [29,30,37]. We found that if we take a non-normalizable KK mode with mass $m^2=(2\mu-1)H^2$, the contribution from the graviscalar in the bulk corresponds to dark radiation at large scales. We also checked that this solution corresponds to the excitation of a small black hole in the bulk by calculating δE_{tt}. It was shown that the asymptotic behavior of δE_{tt} induced by the KK mode with mass $m^2=(2\mu-1)H^2$ precisely agrees with that derived from the black hole solution with a small black hole mass.

The perturbation of the small black hole mass breaks down approaching the black hole. We need to include the nonlinear effect. Thus we expect that non-normalizability of the mode is cured if we take into account the nonlinear effect. For the homogeneous and isotropic cases, we know the resultant nonlinear solution, that is, the black hole spacetime. A nontrivial result here is that the KK mode also induces an anisotropic component of $E_{\mu\nu}$, which vanishes in the long-wavelength limit. This anisotropic component has the same y dependence as δE_{tt}. Thus in order to discuss the bulk geometry at large Hy with anisotropy, it is necessary to find a nonlinear solution with anisotropy. This deserves further investigation.

Now we discuss the behavior of $\delta\pi_F$ related to dark radiation, which plays an important role when we compute CMB anisotropy in brane world models. As shown in Sec. IV B 2, $\delta\pi_F$ that originates from the perturbatively small black hole mass in the bulk is related to $\delta\rho_F$ at large scales as

$$\delta\pi_F=\frac{(-k\eta)^2}{4\mu(\mu+1)}\delta\rho_F. \qquad (81)$$

Let us compare this result with the one obtained in Ref. [28] in the RS two-brane model by solving the bulk geometry using a low energy approximation [40]. (It should be noted that we are assumed to live on the positive tension brane.) In Ref. [28], the matter on the brane is assumed to have the equation of state $P=w\rho$. If we consider the four-dimensional cosmological constant as the matter on the brane ($w=-1$), the model in Ref. [28] corresponds to our model with $b=0$ except for the existence of the second brane. If the distance between two branes is constant, the solution for an anisotropic component of $E_{\mu\nu}$ becomes

$$\kappa_4^2\delta\pi_F=\frac{1}{5}k^2a^{-2}\frac{\delta\rho_F}{\rho} \qquad (82)$$

$$=\frac{1}{5}k^2a^{-2}\frac{\delta C_r a^{-4}}{\rho}, \qquad (83)$$

where a is the cosmic scale factor and $\kappa_4^2\rho=3H^2$. δC_r is the integration constant associated with the perturbation of the radion, which is defined as the physical distance between two branes. If we rewrite the above relation using the conformal time, the result (83) becomes

$$\delta\pi_F=\frac{(-k\eta)^2}{15}\delta\rho_F. \qquad (84)$$

This relation is exactly the same as Eq. (81) for $b=0$ ($\mu=3/2$). It is a somewhat surprising result. Despite the fact that the low energy expansion scheme is applicable only for the two-brane model, our result shows that it can be used to investigate the bulk gravitational field in the one-brane model if we choose the boundary condition at the second brane properly. We plan a more detailed study on the effectiveness of the low energy expansion scheme using our exactly solvable model [41].

In this paper, we did not consider the normalization of the perturbations. This can be fixed if we perform the quantization of the perturbations. This was partially done in Ref. [29] for the scalar field perturbation. However, precisely speaking, we need to quantize two degrees of freedom independently. This issue is also left for a future study [41].

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APPENDIX A: PERTURBED EINSTEIN EQUATIONS ON THE BRANE

In this appendix, we present the perturbed effective four-dimensional Einstein equations. The linear scalar metric and scalar field in the longitudinal gauge is taken as

$$ds^2=-[1+2\Phi(t)Y]dt^2+e^{2\alpha}[1+2\Psi(t)Y]\delta_{ij}dx^i dx^j,$$
$$\varphi=\varphi+\delta\varphi Y. \tag{A1}$$

The perturbed four-dimensional Einstein equations are given as follows. For (t,t),

$$6\dot{\alpha}^2\left(\Phi-\frac{\dot{\Psi}}{\dot{\alpha}}\right)-2k^2e^{-2\alpha}\Psi$$
$$=-\delta\rho_F-3\sqrt{2}b\kappa\dot{\alpha}\dot{\varphi}\left(\Phi-\frac{\dot{\Psi}}{\dot{\alpha}}\right)+\sqrt{2}b\kappa k^2e^{-2\alpha}\delta\varphi$$
$$-2\sqrt{2}b\kappa\frac{\Delta+4}{\Delta}\lambda_0^2\delta e^{-2\sqrt{2}b\kappa\varphi}\delta\varphi. \tag{A2}$$

For (i,i),

$$\ddot{\Psi}+3\dot{\alpha}\dot{\Psi}-2\Phi\ddot{\alpha}-3\dot{\alpha}^2\Phi-\dot{\alpha}\dot{\Phi}+\frac{1}{3}k^2e^{-2\alpha}(\Psi+\Phi)$$
$$=-\sqrt{2}b\kappa\left((\dot{\Psi}-2\dot{\alpha}\Phi)\dot{\varphi}-\ddot{\varphi}\Phi-\frac{1}{2}\dot{\varphi}\dot{\Phi}+\frac{1}{2}\delta\ddot{\varphi}\right.$$
$$\left.+\dot{\alpha}\delta\dot{\varphi}+\frac{1}{3}k^2e^{-2\alpha}\delta\varphi\right)+\frac{\Delta+11/3}{2}\kappa^2(\dot{\varphi}^2\Phi-\dot{\varphi}\delta\dot{\varphi})$$
$$+\sqrt{2}b\kappa\frac{\Delta+4}{\Delta}\lambda_0^2\delta e^{-2\sqrt{2}b\kappa\varphi}\delta\varphi$$
$$-\frac{1}{6}(\delta\rho_F+3\sqrt{2}b\kappa\delta F_\varphi). \tag{A3}$$

For (i,j),

$$-k^2e^{-2\alpha}(\Phi+\Psi+\sqrt{2}b\kappa\delta\varphi)=\delta\pi_F. \tag{A4}$$

For (t,i),

$$-2ke^{-\alpha}(\dot{\Psi}-\dot{\alpha}\Phi)=\frac{2\kappa^2}{3}ke^{-\alpha}\dot{\varphi}\delta\varphi+\sqrt{2}b\kappa ke^{-\alpha}$$
$$\times(\delta\dot{\varphi}+\sqrt{2}b\kappa\dot{\varphi}\delta\varphi-\dot{\varphi}\Phi)-\delta q_F. \tag{A5}$$

Introducing a canonical variable for scalar perturbations,

$$Q=\delta\varphi-\frac{\dot{\varphi}}{\dot{\alpha}}\Psi=\delta\varphi-\frac{3\sqrt{2}b}{\kappa}\Psi, \tag{A6}$$

the perturbed scalar field equation can be rewritten as

$$\ddot{Q}+(3\dot{\alpha}+\sqrt{2}b\kappa\dot{\varphi})\dot{Q}+k^2e^{-2\alpha}Q$$
$$=-\sqrt{2}b\kappa^{-1}(\delta\rho_F+\delta\pi_F)-\delta F_\varphi. \tag{A7}$$

Here we used the scalar field equation (25) and the four-dimensional Einstein equations (A2), (A3), and (A4). It should be noted that Q is related to the curvature perturbation as

$$R_c=\frac{\dot{\alpha}}{\dot{\varphi}}Q, \tag{A8}$$

which affects the amplitude of the CMB anisotropy.

The above Einstein equations and the equation for Q are not closed but include the terms due to the KK modes. As shown in the Sec. IV A, the constraint equations (27) for $\delta F_{\mu\nu}$ and δF_φ are not sufficient to determine these variables. We must solve the bulk dynamics to completely understand cosmological perturbations on the brane.

APPENDIX B: SOLUTIONS IN THE LONGITUDINAL GAUGE

In this appendix, we present the solutions of five-dimensional Einstein equations for scalar perturbation in the longitudinal gauge. By a gauge transformation of the solutions given in Sec. IV B 1 to the longitudinal gauge, we get

$$\Psi_L=-\frac{2}{3(\Delta+2)}\frac{i\nu+\mu-1}{i\nu-1}c_1\left[\frac{\Delta+2}{\Delta+4}(i\nu+\mu)(i\nu-\mu)\varrho(\eta)\psi_m-\varrho(\eta)\frac{\cosh Hy}{\sinh Hy}\frac{\psi_m'}{H}\right.$$
$$\left.+\frac{2}{\Delta+4}(i\nu-1)(i\nu+\mu)\frac{\rho_{i\nu}}{(-k\eta)^2}\zeta_m(y)\right]+\frac{3\Delta+8}{3(\Delta+4)}c_2\left[\frac{i\nu+\mu-1}{i\nu-1}\varrho(\eta)\psi_m+\frac{2}{\Delta+2}\frac{\rho_{i\nu}}{(-k\eta)^2}\zeta_m(y)\right], \tag{B1}$$

$$\Phi_L = \frac{2}{3(\Delta+4)}(i\nu+\mu)(i\nu+\mu-1)c_1\Bigg[-\frac{1}{(i\nu+\mu)(i\nu-1)}\varrho(\eta)\Bigg((i\nu+\mu)(i\nu-\mu)\psi_m - \frac{\Delta+4}{\Delta+2}\frac{\cosh Hy}{\sinh Hy}\frac{\psi_m'}{H}\Bigg)$$
$$-\frac{2}{\Delta+2}\frac{\rho_{i\nu}}{(-k\eta)^2}\frac{\cosh Hy}{\sinh Hy}\frac{\psi_m'}{H} + (i\nu-\mu)(3i\nu+3-\mu)\frac{\rho_{i\nu}}{(-k\eta)^2}\psi_m\Bigg] - \frac{3\Delta+8}{3(\Delta+4)}c_2\Bigg[\Bigg(-\rho_{i\nu}+\frac{4\mu-3}{2(i\nu-1)}$$
$$\times(\rho_{i\nu}+\rho_{i\nu-2}) + (i\nu-\mu)(3i\nu+3-\mu)\frac{\rho_{i\nu}}{(-k\eta)^2}\Bigg)\psi_m - \frac{2}{\Delta+2}\frac{\rho_{i\nu}}{(-k\eta)^2}\frac{\cosh Hy}{\sinh Hy}\frac{\psi_m'}{H}\Bigg], \tag{B2}$$

$$N_L = -\Psi_L - \Phi_L, \tag{B3}$$

$$A_L = 2k^{-1}e^{\alpha}\psi_m'\Bigg[c_1 k\eta\mu\frac{i\nu+\mu-1}{i\nu-1}(\rho_{i\nu}+\rho_{i\nu-2})$$
$$-\frac{3\Delta+8}{\Delta+4}\Bigg(c_2 - \frac{2c_1}{3\Delta+8}(i\nu+\mu)(i\nu+\mu-1)\Bigg)\frac{1}{k\eta}(k\eta\rho_{i\nu-1} + (i\nu-\mu+1)\rho_{i\nu})\Bigg], \tag{B4}$$

$$\delta\varphi_L = 3\sqrt{2}b\kappa^{-1}(-c_2\rho_{i\nu}+\Psi_L), \tag{B5}$$

where

$$\varrho(\eta) = \rho_{i\nu} + \frac{\mu}{i\nu+\mu-1}\rho_{i\nu-2}, \tag{B6}$$

$$\zeta_m(y) = (i\nu-\mu)\psi_m + \frac{\cosh Hy}{\sinh Hy}\frac{\psi_m'}{H}. \tag{B7}$$

If we take

$$c_1 = \frac{c_2(3\Delta+8)}{2(i\nu+\mu)(i\nu+\mu-1)}, \tag{B8}$$

this solution becomes the one already obtained by Koyama and Takahashi [29,30].

APPENDIX C: PERTURBATION FORMULAS FOR 5D WEYL TENSOR

In this appendix, we give the Wey tensor in our background spacetime. Here we take the Gaussian normal gauge condition (34). The explicit expressions are as follows:

$$C = \frac{e^{2W}}{2}[\Phi''-\Psi''] + \frac{e^{2W+2\sqrt{2}b\kappa\varphi}}{2}\Bigg[-\ddot{N}+\ddot{\Psi}-\frac{k}{3}\dot{B}+\sqrt{2}b\kappa\dot{\varphi}\Bigg(\dot{\Phi}+\dot{\Psi}-\frac{k}{3}B-2\dot{N}\Bigg)+\dot{\alpha}(\dot{N}-\dot{\Phi})$$
$$-2N(\sqrt{2}b\kappa\ddot{\varphi}+2b^2\kappa^2\dot{\varphi}^2-\ddot{\alpha}-\sqrt{2}b\kappa\dot{\varphi}\dot{\alpha})\Bigg] + \frac{k^2}{6}e^{2W+2\sqrt{2}b\kappa\varphi-2\alpha}\Bigg[N+\Phi-2\Psi-\frac{2}{3}E\Bigg], \tag{C1}$$

$$C_2 = -\frac{2}{3}e^{2W}E'' + \frac{k^2}{3}e^{2W+2\sqrt{2}b\kappa\varphi-2\alpha}\Bigg[-2N+\Psi+\Phi+\frac{1}{3}E\Bigg]$$
$$-\frac{1}{3}e^{2W+2\sqrt{2}b\kappa\varphi}[\ddot{E}+k\dot{B}-(\dot{E}+kB)(2\sqrt{2}b\kappa\dot{\varphi}-3\dot{\alpha})+E(\sqrt{2}b\kappa\ddot{\varphi}+2b^2\kappa^2\dot{\varphi}^2-\ddot{\alpha}-\sqrt{2}b\kappa\dot{\varphi}\dot{\alpha})], \tag{C2}$$

where we expanded $\delta^{(5)}C_{\mu y \nu y}$ in terms of the scalar harmonics as

$$\delta^{(5)}C_{tyty} = CY,$$
$$\delta^{(5)}C_{iyjy} = e^{2\alpha}\Bigg(\frac{1}{3}CY\delta_{ij} + C_2 Y_{ij}\Bigg). \tag{C3}$$

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[3] K.i. Maeda and M. Sasaki, Suppl. Prog. Theor. Phys. **148**, (2002).
[4] A. Lukas, B.A. Ovrut, and D. Waldram, Phys. Rev. D **60**, 086001 (1999).
[5] K.i. Maeda and D. Wands, Phys. Rev. D **62**, 124009 (2000).
[6] S. Nojiri, O. Obregon, and S.D. Odintsov, Phys. Rev. D **62**, 104003 (2000); S. Nojiri, O. Obregon, S.D. Odintsov, and V.I. Tkach, *ibid.* **64**, 043505 (2001).
[7] M. Cvetič, H. Lü, and C.N. Pope, Phys. Rev. D **63**, 086004 (2001).
[8] V. Bozza, M. Gasperini, and G. Veneziano, Nucl. Phys. **B619**, 191 (2001).
[9] O. Seto and H. Kodama, Phys. Rev. D **63**, 123506 (2001).
[10] D. Langlois and M. Rodríguez-Martínez, Phys. Rev. D **64**, 123507 (2001).
[11] H. Ochiai and K. Sato, Phys. Lett. B **503**, 404 (2001).
[12] S. Kobayashi and K. Koyama, J. High Energy Phys. **12**, 056 (2002).
[13] D. Langlois and M. Sasaki, Phys. Rev. D **68**, 064012 (2003).
[14] M. Minamitsuji, Y. Himemoto, and M. Sasaki, Phys. Rev. D **68**, 024016 (2003).
[15] S. Kanno and J. Soda, Gen. Relativ. Gravit. **36**, 689 (2004).
[16] T. Shiromizu, K.-i. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[17] D. Ida, J. High Energy Phys. **09**, 014 (2000); P. Kraus, *ibid.* **12**, 011 (1999); S. Mukohyama, Phys. Lett. B **473**, 241 (2000).
[18] K. Ichiki, M. Yahiro, T. Kajino, and M. Orito, Phys. Rev. D **66**, 043521 (2002).
[19] R. Maartens, astro-ph/0402485 and references therein.
[20] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000).
[21] H. Kodama, A. Ishibashi, and O. Seto, Phys. Rev. D **62**, 064022 (2000).
[22] R. Maartens, Phys. Rev. D **62**, 084023 (2000).
[23] D. Langlois, Phys. Rev. D **62**, 126012 (2000).
[24] C. van de Bruck, M. Dorca, R.H. Brandenberger, and A. Lukas, Phys. Rev. D **62**, 123515 (2000).
[25] K. Koyama and J. Soda, Phys. Rev. D **62**, 123502 (2000); K. Koyama, *ibid.* **66**, 084003 (2002).
[26] K. Koyama and J. Soda, Phys. Rev. D **65**, 023514 (2002); J. Soda and K. Koyama, Suppl. Prog. Theor. Phys. **148**, 136 (2002).
[27] D. Langlois, R. Maartens, M. Sasaki, and D. Wands, Phys. Rev. D **63**, 084009 (2001).
[28] K. Koyama, Phys. Rev. Lett. **91**, 221301 (2003).
[29] K. Koyama and K. Takahashi, Phys. Rev. D **67**, 103503 (2003).
[30] K. Koyama and K. Takahashi, Phys. Rev. D **68**, 103512 (2003).
[31] S. Kobayashi, K. Koyama, and J. Soda, Phys. Lett. B **501**, 157 (2001).
[32] Y. Himemoto and M. Sasaki, Phys. Rev. D **63**, 044015 (2001).
[33] J. Yokoyama and Y. Himemoto, Phys. Rev. D **64**, 083511 (2001).
[34] N. Sago, Y. Himemoto, and M. Sasaki, Phys. Rev. D **65**, 024014 (2002).
[35] Y. Himemoto, T. Tanaka, and M. Sasaki, Phys. Rev. D **65**, 104020 (2002).
[36] Y. Himemoto and T. Tanaka, Phys. Rev. D **67**, 084014 (2003); T. Tanaka and Y. Himemoto, *ibid.* **67**, 104007 (2003).
[37] T. Kobayashi and T. Tanaka, Phys. Rev. D **69**, 064037 (2004).
[38] M. Minamitsuji and M. Sasaki, Phys. Rev. D (to be published), gr-qc/0312109.
[39] R.G. Cai, J.Y. Ji, and K.S. Soh, Phys. Rev. D **57**, 6547 (1998); H.A. Chamblin and H.S. Reall, Nucl. Phys. **B562**, 133 (1999).
[40] S. Kanno and J. Soda, Phys. Rev. D **66**, 083506 (2002); T. Shiromizu and K. Koyama, *ibid.* **67**, 084022 (2003).
[41] H. Yoshiguchi and K. Koyama, in preparation.